Estimation of an effective Rayleigh number for convection in a vertically inhomogeneous porous medium or clear fluid

D. A. Nield
Department of Engineering Science, University of Auckland, Auckland, New Zealand

The concept of an effective Rayleigh number (for a clear fluid), or effective Rayleigh-Darcy number (for a porous medium), is introduced in the context of convection in a vertically inhomogeneous horizontal layer. Estimates for this quantity, based on mean values of the physical quantities, are proposed. These estimates are compared with known, accurate results.

Keywords: effective Rayleigh number; Rayleigh-Bénard convection

1. Introduction

In the introduction of their recent paper, Leu and Jang (1993, p. 203) (see also Jang and Leu 1993) wrote

> Kassoy and Zebib (1975) examined the variable viscosity effects on the onset of convection in a water-saturated porous medium. The critical Rayleigh number is found to be substantially reduced from the classical value of $4\pi^2$. Straus and Schubert (1977) and Horne and O'Sullivan (1978) have also considered the onset of convection of water as a non-Boussinesq fluid with viscosity and thermal expansivity dependence. The critical Rayleigh number is reduced by as much as a factor of 31 below the classical value of $4\pi^2$.

From this, the reader may be given the impression that property variation can have a destabilizing effect. Indeed, in their own paper, which is concerned with vortex instability in a free-convection boundary-layer flow, Leu and Jang (1993, p. 203) reported that the "numerical results indicate that the variable viscosity effect enhances the heat transfer and destabilizes the flow." This is misleading. It is just a reduction in viscosity relative to a fixed value that is involved in their paper. Common sense indicates that if the Rayleigh number is redefined in terms of some suitable intermediate viscosity, then the numerical value of the critical Rayleigh should be unaltered, so there is no destablization per se. This raises the questions of whether the intermediate viscosity can be estimated by some appropriate mean value (some authors have used the arithmetic mean and others have used the viscosity coresponding to the midplane temperature) and the degree to which the distribution about its mean value affects the criterion for instability.

The present investigation is an attempt to answer these and other questions. It is concerned with convection induced by vertical temperature gradients in either a porous medium or a clear (of solid material) fluid. For flow in porous media, relevant papers (in addition to those mentioned earlier) include those by Morland et al. (1977), Blythe and Simpkins (1981), Patil and Vaidyanathan (1981 and 1982), Or (1989) on viscosity variation, and the studies of layered porous media reviewed by McKibbin (1985) and Nield and Bejan (1992, Section 6.13). For flow in clear fluids, the literature is extensive. Of particular relevance to the effects of viscosity variation are the pioneering studies of Palm (1960) and Jenssen (1963) and the comprehensive study by Stengel et al. (1982).

This investigation is directed to the broad question of general property variation but, because of the availability of previous studies, viscosity variation is emphasized later in this paper. The investigation is guided by analytic results obtained from a two-layer model. This discrete model provides a rough analogy to the continuous situation. The two-layer model was chosen because it is adequate to reveal the complexity of the interaction between the various property variations, while at the same time it is sufficiently simple for general conclusions to be drawn from the results. (A three-layer model for a clear fluid leads to an excessive amount of algebra. The author has carried out an analysis of the three-layer model for a porous medium, but the results are so complicated that they do not provide significant additional aid for the task of estimating an effective Rayleigh-Darcy number for a general situation.)

2. A two-layer model

We consider Bénard convection in two superimposed horizontal layers of either porous media or clear fluids. (For the porous media, Darcy's law is assumed to hold.) Subscripts 1 and 2 refer to the lower and upper layers, respectively, of depths d_1 and d_2. The lower boundary is uniformly heated and rigid. We consider, separately, the cases in which the upper boundary is rigid or free. In each case, the perturbation heat flux at both boundaries is taken to be zero; this ensures that the critical horizontal wave number is zero and permits a simple analytical solution. Standard linear stability analysis,

Address reprint requests to Professor Nield at the Department of Engineering Science, University of Auckland, Private Bag 92019, Auckland, New Zealand.

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following the procedure described by Nield (1977), leads to the following criteria for the onset of convection. Here

$E_m = g\alpha K/\nu$, $F_m = k_m\beta/\kappa_m$

2.1. Porous media

Case 1a. porous media; both boundaries impermeable:

$$[E_{m1}F_{m1}(d_1^4 + 4d_1^3d_2) + 3(E_{m1}F_{m2} + E_{m2}F_{m1})d_1^2d_2^2 + E_{m2}F_{m2}(4d_1d_2^4 + d_2^4)]/[(k_{m1}d_1 + k_{m2}d_2)(d_1 + d_2)] = 12 \quad (1)$$

Case 1b. porous media; lower boundary impermeable, upper boundary at zero perturbation pressure:

$$[2E_{m1}F_{m1}d_1^3 + 3(E_{m1}F_{m2} + E_{m2}F_{1m})d_1^2d_2 + E_{2m}F_{2m}(6d_1d_2^3 + 2d_2^3)]/2(k_{m1}d_1 + k_{m2}d_2) = 3 \quad (2)$$

The expressions on the left-hand side of Equations 1 and 2 are effective Rayleigh-Darcy numbers. For the homogeneous case ($\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$, $\kappa_{m1} = \kappa_{m2} = \kappa_m$, $\nu_1 = \nu_2 = \nu$, $k_{m1} = k_{m2} = k_m$, $K_1 = K_2 = K$), they each reduce to $g\alpha\beta Kd^2/\kappa_m\nu$, where $d = d_1 + d_2$, so that Equation 1 gives $R_m = 12$ and Equation 2 gives $R_m = 3$. These, as expected, are the values of the critical Rayleigh-Darcy numbers found by Nield (1968).

Clearly, the general case is complicated. In particular, there is a coupling between thermal quantities grouped into the F_{mi} factors and the hydrodynamic quantities grouped into the E_{mi} factors. Obviously, there is no simple exact universal formula for the effective Rayleigh-Darcy number in a general situation. In fact, the true effective Rayleigh number depends on the distribution of the varying property or properties. The aim here is to provide a useful estimate (which is independent of the particular distribution) of this quantity.

Consider, for simplicity, the situation when $d_1 = d_2 = d/2$. (Conclusions will be drawn that are, it is believed, qualitatively independent of this simplification.) Consider first Case 1a, an example of symmetric boundary conditions (same conditions at the top as at the bottom for the perturbation problem). For the case of thermal homogeneity ($\beta_1 = \beta_2 = \beta$, $\kappa_{m1} = \kappa_{m2} = \kappa_m$, $k_{m1} = k_{m2} = k_m$, so that $F_{m1} = F_{m2}$) the correct two-layer, effective Rayleigh-Darcy number is obtained if $(E_{m1} + E_{m2})/2$ is taken as the value of the group E_m in the Rayleigh-Darcy number definition. This suggests that, in a general situation, the best choice of average for E_m is the arithmetic mean (E_{mA}) for the case of symmetric boundary conditions. In particular, if the viscosity ν is the only inhomogeneous quantity, then the harmonic mean (ν_H) should be the best choice for average viscosity. In general, an estimate for E_A is $g\alpha_AK_A/\nu_H$. Similarly, for the case of hydrodynamic homogeneity ($\alpha_1 = \alpha_2 = \alpha$, $\nu_1 = \nu_2 = \nu$, $K_1 = K_2 = K$, so that $E_{m1} = E_{m2}$), Equation 1 suggests that the best choice of F_m/k_m is F_{mA}/k_{mH}. Continuity of heat flux requires that $k\beta$ = constant, and this, together with the fact that $\Delta T = \beta_A d$, implies that $F_{mA} = (k_{mH}/\kappa_{mH})\Delta T/d$. It follows that an estimate for the Rayleigh number R_m is given by

$$R_{mest} = \frac{g\alpha_A K_A d\Delta T}{\nu_H \kappa_{mH}} \quad (3)$$

The reader will note that arithmetic means of quantities appear in the numerator of this expression and harmonic means in the denominator.

In order to estimate the order of accuracy involved in the estimation, the author has investigated the case of a single layer of porous medium, homogeneous except for viscosity variation, between impermeable conducting boundaries, using the same, second-order Galerkin approximation employed by Nield (1990). The results showed that the estimated Rayleigh-Darcy number based on the harmonic mean of the kinematic viscosity (as in Equation 3) is independent of the slope of the viscosity variation function, but is dependent on its curvature. In fact, the estimated Rayleigh-Darcy number is too large by a factor $1 + \delta$, where an estimate of the quantity δ (assumed to be small) is given by

$$\delta \approx \frac{\nu d^2}{42}\frac{d^2(\nu^{-1})}{dz^2}$$

Now consider Case 1b, again with $d_1 = d_2 = d/2$. For the thermally homogeneous case, the left-hand side of Equation 2 reduces to $(\frac{5}{16}E_{m1} + \frac{11}{16}E_{m2})F_md^2/k_m$. Now a weighted arithmetic mean is involved, the layer nearer the less restrictive boundary having the greater weight. For the hydrodynamically homogeneous case, the same expression reduces to $E_m(\frac{5}{16}F_{m1} + \frac{11}{16}F_{m2})d^2/k_m$.

This suggests that R_{mest} be obtained as before, using Equation 5, but now the various means being obtained using an appropriate weighting factor. For Case 1b, the weighting factor is 11:5. This suggests that in the general case, with nonsymmetric boundaries, a weighting factor on the order of 2:1 may be suitable, but since this factor depends on the boundary conditions, further investigation is needed for the selection of the best value.

Notation

d	Layer depth (m)
E	$g\alpha/\nu$ (m^{-1} s^{-1} K^{-1})
E_m	$g\alpha K/\nu$ (m s^{-1} K^{-1})
F	$k\beta/\kappa$ (W m^{-4} s)
F_m	$k\beta/\kappa_m$ (W m^{-4} s)
g	Gravitational acceleration (m s^{-2})
k	Thermal conductivity (W m^{-1} K^{-1})
R	Rayleigh number ($g\alpha\beta d^4/\kappa\nu$)
R_m	Rayleigh-Darcy number ($g\alpha\beta Kd^2/\kappa_m\nu$)
T	Temperature (K)
z	Vertical coordinate (m)

Greek symbols

α	Volume expansion coefficient (K^{-1})
β	Temperature gradient (m^{-1} K)
δ	Fractional error
Δ	Difference across the layer
ε	Viscosity variation amplitude
κ	Thermal diffusivity (m^2 s^{-1})
μ	Dynamic viscosity (kg m^{-1} s^{-1})
ν	Kinematic viscosity (μ/ρ), (m^2 s^{-1})
ρ	Density (kg m^{-3})

Subscripts

1	Lower layer
2	Upper layer
A	Arithmetic mean
G	Geometric mean
H	Harmonic mean
m	Porous medium

2.2. Clear fluids

Using a two-layer model, the following criteria for neutral stability are obtained. Here, $E = g\alpha/\nu$ and $F = k\beta/\kappa$.

Case 2a. clear fluids; both boundaries rigid:

$$\begin{aligned}&[E_1F_1\{\mu_2/\mu_1)d_1^9 + 9d_1^8d_2 + 36d_1^7d_2^2 + 64d_1^6d_2^3 + 36(\mu_1/\mu_2)d_1^5d_2^4\}\\ &\quad + E_1F_2\{10d_1^6d_2^3 + 45d_1^5d_2^4 + 45(\mu_1/\mu_2)d_1^4d_2^5 + 36(\mu_1/\mu_2)d_1^3d_2^6\}\\ &\quad + E_2F_1\{10(\mu_2/\mu_1)d_1^6d_2^3 + 45(\mu_2/\mu_1)d_1^5d_2^4 + 45d_1^4d_2^5 + 10d_1^3d_2^6\}\\ &\quad + E_2F_2\{36(\mu_2/\mu_1)d_1^4d_2^5 + 64d_1^3d_2^6 + 36d_1^2d_2^7 + 9d_1^1d_2^8\\ &\quad + (\mu_1/\mu_2)d_2^9\}]/[(k_1d_1 + k_2d_2)\{(\mu_2/\mu_1)d_1^4 + 4d_1^3d_2 + 6d_1^2d_2^2\\ &\quad + (\mu_1/\mu_2)d_2^4\}] = 720 \end{aligned}\tag{4}$$

Case 2b. clear fluids; lower boundary rigid, upper boundary free:

$$\begin{aligned}&[E_1F_1\{3d_1^8 + 24d_1^7d_2 + 64d_1^6d_2^2 + 48(\mu_1/\mu_2)d_1^5d_2^3\}\\ &\quad + E_1F_2\{10d_1^6d_2^2 + 60d_1^5d_2^3 + 75(\mu_1/\mu_2)d_1^4d_2^4 + 20(\mu_1/\mu_2)d_1^5d_2^3\}\\ &\quad + E_2F_1\{10(\mu_2/\mu_1)d_1^6d_2^2 + 60(\mu_2/\mu_1)d_1^5d_2^3 + 75d_1^4d_2^4 + 20d_1^3d_2^5\}\\ &\quad + E_2F_2\{60(\mu_2/\mu_1)d_1^4d_2^4 + 128d_1^3d_2^5 + 84d_1^2d_2^6 + 27d_1d_2^7\\ &\quad + 3(\mu_1/\mu_2)d_2^8\}]/[3(k_1d_1 + k_2d_2)\{d_1^3 + 3d_1^2d_2 + 3d_1^2d_2^2\\ &\quad + (\mu_1/\mu_2)d_2^3\}] = 320 \end{aligned}\tag{5}$$

For the homogeneous situation, the expressions on the left-hand side of Equations 4 and 5 each reduce to the Rayleigh number R as usually defined and so, in general, they can be regarded as effective Rayleigh numbers. The values 720 and 320 are well-known critical values—they were first obtained by Sparrow et al. (1964).

Now the viscosity ratio appears explicitly, as well as being involved implicitly, via E_1 and E_2. From now on, it will be assumed that μ_2/μ_1 is sufficiently close to unity so that the explicit dependence (which involves only certain terms in a sum of terms) is relatively unimportant. Consider the case where $d_1 = d_2 = d/2$. The form to which Equation 4 reduces suggests that, for the general case with symmetric boundary conditions, one should estimate E, F and k by their arithmetic means, and, consequently, estimate R by

$$\mathrm{R}_{\mathrm{est}} = \frac{g\alpha_A\, d^3\Delta T}{\nu_H\kappa_H}\tag{6}$$

On the other hand, when $d_1 = d_2 = d/2$, the left-hand side of Equation 5 reduces, for the thermally homogeneous case, to $(\frac{19}{48}E_1 + \frac{29}{48}E_2)Fd^4/k$ and, for the hydrodynamically homogeneous case, to $E(\frac{19}{48}F_1 + \frac{29}{48}F_2)d^4/k$. This suggests that in the general case with nonsymmetric boundary conditions, a weighting factor of the order of 3:2 may be suitable.

3. Comparison with known results

The pioneering work on the effect of viscosity variation on Rayleigh-Bénard convection is by Palm (1960). He treated the idealized case where the kinematic viscosity varies with vertical coordinate z according to $\nu = \nu_0(1 + \varepsilon\cos\pi z)$, for $0 \le z \le 1$, where ν_0 and ε are positive constants, $\varepsilon \ll 1$. Palm considered the case of conducting stress-free boundaries. He found that R_0, the critical Rayleigh number based on the kinematic viscosity ν_0, varied with ε according to $\mathrm{R}_0 = 27\pi^4/4(1 - 0.259\varepsilon^2)$. This means that the effective Rayleigh number is $\mathrm{R}_0/(1 - 0.259\varepsilon^2) \approx \mathrm{R}_0(1 + 0.259\varepsilon^2)$. The estimated effective Rayleigh number based on the arithmetic mean ν_A is just R_0. The estimated effective Rayleigh number according to Equation 6 (i.e., that based on the harmonic mean ν_H, which is found to be $\nu_0(1 - \varepsilon^2)^{1/2}$) is $\mathrm{R}_0/(1 - \varepsilon^2)^{1/2} \approx \mathrm{R}_0(1 + 0.5\varepsilon^2)$. We see that using ν_A gives a result that is too small and using ν_H gives (as expected) a result that is too large. One is led to speculate that using the geometric mean ν_G might give a better result. Indeed, since in this case $\nu_G = \nu_0(1 - \varepsilon^2/2)^{1/2}$, the estimated effective Rayleigh number is approximately $\mathrm{R}_0(1 + 0.25\varepsilon^2)$, in very good agreement with the results of Palm.

Jenssen (1963) repeated the analysis of Palm, using the same viscosity variation, but now for the cases of (i) two rigid conducting boundaries, and (ii) one rigid conducting boundary and one free conducting boundary. For case (i), he found the critical Rayleigh number to be given by $\mathrm{R}_0 = 1708(1 - 0.247\varepsilon^2)$, which means that the effective Rayleigh number is $\mathrm{R}_0/(1 - 0.247\varepsilon^2) \approx \mathrm{R}_0(1 + 0.247\varepsilon^2)$. This agrees remarkably well with the estimate $\mathrm{R}_0(1 + 0.25\varepsilon^2)$ based on ν_G. For case (ii), he found the critical Rayleigh number to be given by $\mathrm{R}_0 = 1100.6(1 - 0.362\varepsilon)$, which means that the effective Rayleigh number is $\mathrm{R}_0/(1 - 0.362\varepsilon) \approx \mathrm{R}_0(1 + 0.362\varepsilon)$. A weighting ratio of 3:2 and the harmonic means for the two layers of depth $d/2$ leads to the estimate $\mathrm{R}_0(1 + 0.127\varepsilon)$. The same weighting ratio, but with geometric means, leads to the same result, to the first order in ε. In order to reproduce Jenssen's result, one must use 3.64:1 as the weighting ratio.

The adaptation of the Palm-Jenssen theory to a porous medium (rather than a clear fluid) was made by Patil and Vaidyanathan (1981). Their results for the critical Rayleigh number are given in column 3 of Table 1. In column 4, we have listed the calculated value of the constant c defined by

$$\mathrm{R} = \mathrm{R}_0(1 - c\varepsilon^2)\tag{7}$$

The table shows that their results can be fitted by Equation 7 and that c varies from about 0.188 (compared with the value 0.193 obtained by Palm [1960]) in the clear fluid limit $(K/d^2 \to \infty)$ to about 0.266 in the Darcy limit $(K/d^2 \to 0)$. The estimate given by Equations 3 or 6 would lead to Equation 7 with $c = 0.5$ (if a harmonic mean value for the viscosity is used) or $c = 0.25$ (if a geometric mean value for the viscosity is used), independent of the value of K/d^2. [The fact that Patil and Vaidyanathan presented results for $\varepsilon = 0.5$, which is formally inconsistent with the assumption that ε is small compared with unity, does not affect the present comparison.]

Unfortunately, the results of Kassoy and Zebib (1975) cannot be used for comparison here because they appear to be anomalous in two respects. Firstly, they predict that the critical

Table 1 Analysis of the results of Patil and Vaidyanathan (1981). Their calculated values of the critical Rayleigh number are listed in the third column. For each value of the Darcy number K/d^2, R_0 is the corresponding value of R when $\varepsilon = 0$. The Rayleigh-Darcy number R_m is related to R by $\mathrm{R}_m = \mathrm{R}K/d^2$.

K/d^2	ε	R	$(\mathrm{R}_0 - \mathrm{R})/\mathrm{R}_0\varepsilon^2$
∞	0.0	657.511	
	0.1	656.272	0.188
	0.5	626.568	0.188
0.5	0.0	745.443	
	0.1	744.049	0.187
	0.5	710.636	0.187
0.01	0.0	4699.14	
	0.1	4688.35	0.230
	0.5	4431.46	0.228
0.0001	0.0	394785.1	
	0.1	393729.0	0.266
	0.5	368709.7	0.264
0.00001	0.0	3947841	
	0.1	3937274	0.267
	0.5	3686713	0.265

horizontal wave number increases as the amount of viscosity variation (due to change in temperature) increases, whereas all other reports known to the author predict a decrease. Secondly, their results are internally inconsistent because the indicated limit of the Rayleigh number as their parameter τ tends to zero is markedly different from the value for τ equals zero. It is planned that a reexamination of their problem will be the subject of a future report.

The author does not know of any published experimental data that are appropriate for comparison.

4. Conclusion

The concept of an effective Rayleigh number (or Rayleigh-Darcy number) has been introduced in the context of the criterion for the onset of Rayleigh-Bénard convection. Whether the concept is useful for convection at supercritical Rayleigh numbers (in the determination of heat flow, for example) is a subject for future investigation. Provided the variation of a property lies within one order of magnitude, a useful rough-and-ready estimate of an effective Rayleigh number is readily obtained. No claim is made about the situation in which a property varies by several orders of magnitude, as in the experiments reported by Stengel et al. (1982). A large variation may lead to the flow being localized in part of the layer and the effective Rayleigh number as introduced here is a global quantity.

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